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An accelerated life testing problem in which the stress s can take only a fixed, finite number of values is known as accelerated life test with step stress. In general, the item under test is started under the lowest stress first and if an item has not failed until a certain specified time, then it is moved into the next level of stress. This process is continued until the item fails. Thus the life test duration of the item is shortened. A Bayesian formulation of the problem is given in this paper. It is assumed that there are only two stress values under consideration (i) standard use environmental condition (ii) a higher level of stress that is fixed in advance and is the same for all items to be tested. However, the time at which an item on test is taken out of use environment and put under higher stress environment can be chosen by the experimenter subject to a cost structure. We consider the inference and the optimal design problem of when the change the stress as the test progress.

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BAYESIAN DESIGN AND ANALYSIS OF ACCELERATED LIFE TESTING WITH STEP STRESS

by

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1. Introduction

In many problems of life testing, the test process may require an unacceptably long time period for its completion if the test is simply carried out under specified standard stress conditions. In such problems, it is generally possible to run the life test under stresses that are higher than the specified standard in order to accelerate the process and shorten the time to its completion. This process is called accelerated life testing. A few of the classic articles in this area are Epstein [1], Chernoff [2], and Bessler, Chernoff, and Marshall [3]. A standard reference is Mann, Schafer, and Singpurwalla [4], Chapter 9.

In an accelerated life test it is assumed that the lifetime T of an item being tested is a random variable with a distribution function (d.f.) $F(t|\theta)$ that depends on an unknown parameter θ. Furthermore, it is assumed that the parameter θ is related to the stress s under which the test is carried out by a specified function of the form $\theta = \psi(s,\alpha)$, where the unknown parameter α determines the precise relationship between s and θ.

Of central interest in the analysis of data from accelerated life tests is the estimation of the parameters θ and α. There are also many interesting questions related to the design of these tests, such as (i) how many items to put on test, (ii) whether to replace items when they fail, (iii) how to change the stress s as the tests progress, and (iv) when to stop the test. Our main attention in this paper will be focussed on question (iii).

We will consider problems in which the stress s can take only a fixed, finite number of values. A problem of this type is called accelerated life testing with <u>step stress</u>. In fact, for simplicity, we will restrict ourselves to problems in which s can take only two values. These values correspond to (i) the standard environmental conditions under which an item will be used in practice, and (ii) a higher level of stress that is fixed in advance and is the same for all items to be tested. We will assume, however, that the time at which an item on test is taken out of the standard environment and put under stress can be chosen by the experimenter subject to a given cost structure. The development in this paper follows that given in [5], where these tests were called <u>partially</u> accelerated life tests. All of the results that are described in this paper without proof, are derived in [5] under somewhat more general conditions.

Some other articles that pertain to accelerated life testing with step stress are [6] and [7], although these articles do not follow the Bayesian approach to be utilized here. A valuable survey of accelerated life testing problems from the Bayesian perspective is given by Mazzuchi and Singpurwalla [8].

We shall denote the lifetime of an item tested under the standard conditions by the random variable T, and we shall let $F(t|\theta)$ denote the d.f. of T. The value of the parameter θ is unknown and is to be estimated. Suppose that if the item has not failed by some specified time x, then it is switched

to the higher level of stress and the test is continued until the item fails. We assume that the effect of this switch is to multiply the remaining lifetime of the item by some unknown factor $\alpha > 0$. Since the effect of switching to the higher stress level will typically be to shorten the life of the test item, α will usually be less than 1. However, it is not necessary for us to impose this restriction on the models that we will be using here.

To describe the model for this accelerated life test, we shall let Y denote the total lifetime of a test item. Thus, Y is defined by the relation

$$Y = \begin{cases} T & \text{for } T \leq x, \\ x + \alpha(T-x) & \text{for } T > x. \end{cases} \tag{1.1}$$

Since switching to the higher stress level can be regarded as tampering with the ordinary life test, x is called the tampering point, and α is called the tampering coefficient. This model and an application were originally introduced by Goel [9].

We shall assume that an experimenter starts with a sample of n items and subjects them to test in the standard environment. If item i has not failed by some prespecified time x_i, then it is put under the higher stress and the test is continued. If T_i would be the lifetime of item i in the standard environment, then the total lifetime Y_i of item i under this step-stress

life test is given by (1.1). It would be possible to consider problems in which the tampering point x_i for item i is chosen sequentially, after the experimenter has observed whether or not some of the other items have previously failed, but we shall not do so in this paper.

Thus, a sample of n observations $Y_1,\ldots,Y_n$ is obtained on the random variable Y corresponding to preassigned tampering points $x_1,\ldots,x_n$. If the observed value y_i of Y_i is less than the corresponding tampering point x_i, then Y_i is called an untampered observation. Otherwise, Y_i is called a tampered observation. In other words, an untampered observation comes from a test item that failed under the standard conditions, and a tampered observation comes from a test item that failed after it had been switched to the higher-stress level. The statistical problems involved in using the model (1.1) are (i) the estimation of θ and a for given values of the tampering points $x_1,\ldots,x_n$ and (ii) the choice of an optimal design for this estimation, i.e., the selection of the best tampering points.

Throughout the paper we shall assume that the random variable T has an exponential distribution with density

$$f(t|\theta) = \theta e^{-\theta t} \quad \text{for } t > 0 \text{ and } \theta > 0. \tag{1.2}$$

However, the results on optimal design to be presented here will be valid for a somewhat broader class of distributions.

In Section 2 we consider the Bayesian decision-theoretic approach to the estimation of α when the value of θ is known. The results developed in that section are then used in Section 3 to study estimation problems in which both θ and α are unknown. In these sections it is assumed that the tampering points $x_1,\ldots,x_n$ for the n items to be put on test are fixed. In Section 4, the optimal choice of the values of $x_1,\ldots,x_n$ is presented for various types of observational costs. In particular, it is shown that for many cost functions, the optimal design uses only the tampering points $x = 0$ and $x = \infty$, and the number of observations to be taken at each of these values is explicitly derived.

2. Bayes estimation

In this section, we shall begin our study of the estimation problem by assuming that the parameter θ has a known value, say $\theta = \theta_0$, and that we want to estimate the unknown parameter α. We will then use these results in the next section for the case in which θ is unknown.

It is convenient to work with the parameter $\beta = 1/\alpha$, rather than directly with the parameter α itself, and we shall assume that the prior distribution of β is a gamma distribution with parameters r and $s\theta_0$, for which the density is

$$g(\beta) = \frac{(s\theta_0)^r}{\Gamma(r)}\ \beta^{r-1} \exp(-s\theta_0\beta) \qquad (2.1)$$

for $\beta > 0$. These distributions form a conjugate family of prior distributions in this problem (see [10], Chapter 9). In fact, if M denotes the number of tampered observations among $Y_1,\ldots,Y_n$, and A denotes the set of indices $i\in\{1,\ldots,n\}$ for which Y_i is a tampered observation, then it can be shown ([10], p. 166) that the posterior distribution of β given the values of $x_1,\ldots,x_n$ and $Y_1,\ldots,Y_n$ is again a gamma distribution with parameters r_1 and $s_1\theta_0$, where

$$r_1 = r + M \text{ and } s_1 = s + \sum_{i\in A} (Y_i - x_i). \tag{2.2}$$

If there are no tampered observations in the sample, then we obtain no information about the value of α and the posterior distribution of α is the same as the prior.

Since β is a scale parameter, it is reasonable to consider loss functions for its estimation that are invariant under changes in the units of measurement of lifetimes. The following two loss functions have this property:

$$L_1(\hat{\beta},\beta) = \frac{(\hat{\beta}-\beta)^2}{\beta^2} = \left(\frac{\hat{\beta}}{\beta} - 1\right)^2 \tag{2.3}$$

and

$$L_2(\hat{\beta},\beta) = \frac{(\hat{\beta}-\beta)^2}{\hat{\beta}^2} = \left(\frac{\beta}{\hat{\beta}} - 1\right)^2 . \tag{2.4}$$

More generally, we might consider a loss function of the form

$$L(\hat{\beta},\beta) = \beta^k \beta^{\ell}(\hat{\beta}-\beta)^2 \qquad (2.5)$$

for some appropriate choice of the values of k and ℓ. For the sake of being explicit we will restrict consideration in this paper to the loss function L_1 given by (2.3). The loss function L_1 for the estimation of β corresponds to the loss function L_2 for the estimation of $\alpha = 1/\beta$.

It can be shown that if $r > 2$ in the prior distribution of β, then the Bayes estimator of β with respect to the loss function L_1 will be

$$\hat{\beta} = \frac{r_1-2}{s_1\theta_0} \quad . \qquad (2.6)$$

[More generally, $\hat{\beta}$ as given by (2.6) will be the Bayes estimator whenever the data are such that $r_1 = r + M > 2$.] Furthermore, for given tampering points $x_1,\ldots,x_n$, it can be shown that the overall Bayes risk of this estimator, calculated with respect to the joint marginal or predictive distribution of the observations $Y_1,\ldots,Y_n$ is

$$E\left(\frac{1}{r_1-1}\right) = E\left(\frac{1}{r+M-1}\right) . \qquad (2.7)$$

The expectation in (2.7) is calculated with respect to the marginal

distribution of M, which can be found as follows:

Let random variables $\zeta_1,\ldots,\zeta_n$ be defined as

$$\zeta_i = \begin{cases} 1 & \text{if } i\in A, \\ 0 & \text{if } i\in A^c, \end{cases} \tag{2.8}$$

where A^c is the set of indices $i\in\{1,\ldots,n\}$ for which the observation Y_i is <u>not</u> tampered. Then $\zeta_1,\ldots,\zeta_n$ are independent given θ_0, and

$$\Pr(\zeta_i = 1) = \Pr(T > x_i \mid \theta_0) = \exp(-\theta_0 x_i). \tag{2.9}$$

Since

$$M = \sum_{i=1}^{n} \zeta_i , \tag{2.10}$$

it follows that the distribution of M is that of the sum of independent Bernoulli random variables, each with its own probability of success as given by (2.9). In the next section, when the value of θ is unknown, we will have to integrate this distribution over the prior distribution of θ in order to obtain the marginal distribution of M.

3. Estimation with both parameters unknown

Suppose now that both of the parameters ϵ and β are unknown. In this situation, a conjugate family of joint prior distributions for β and θ can be specified as follows:

The conditional prior distribution of β, given θ, is a gamma distribution with parameters r and $s\theta$, and the marginal prior distribution of θ is a gamma distribution with parameters r_0 and s_0.

Under these conditions it can be shown that the joint posterior distribution of β and θ will have this same form and can be specified as follows: The conditional posterior distribution of β, given θ, is a gamma distribution with parameters r_1 and $s_1\theta$, where r_1 and s_1 are given by (2.2), and the posterior distribution of θ is a gamma distribution with parameters r_2 and s_2, where r_2 and s_2 are defined by

$$r_2 = r_0 + n - M \text{ and } s_2 = s_0 + \sum_{i \in A} x_i + \sum_{j \in A^c} Y_j . \qquad (3.1)$$

If all the observations are untampered, then $M = 0$, $s_1 = s$, and $s_2 = s_0 + \sum_{i=1}^{n} Y_i$. If all the observations are tampered, then $M = n$, $s_1 = s + \sum_{i=1}^{n} (Y_i - x_i)$, and $s_2 = s_0 + \sum_{i=1}^{n} x_i$.

It should be noted that this posterior distribution does not depend on the values of the tampering points corresponding to the untampered observations. Hence, it does not depend on the method by which these points were chosen. Furthermore, it is interesting to note that the contribution of each tampered observation Y_j to the posterior distribution of θ is the same as that of an observation censored at x_i in an ordinary life test based on the exponential distribution.

Now consider the estimation of θ in this problem: The same loss functions (2.3), (2.4), and (2.5) that were discussed for the estimation of β will also be appropriate for the estimation of θ. In particular, we shall again use the loss function L_1 given by (2.3), so

$$L_1(\hat{\theta},\theta) = \left(\frac{\hat{\theta}}{\theta} - 1\right)^2 . \tag{3.2}$$

It follows that if $r_0 > 2$ in the prior distribution of θ, then the Bayes estimator will be

$$\hat{\theta} = \frac{r_2-2}{s_2} , \tag{3.3}$$

where r_2 and s_2 are given by (3.1). Furthermore, the Bayes risk of this estimator will be

$$E\left(\frac{1}{r_2-1}\right) = E\left(\frac{1}{r_0+n-M-1}\right) , \tag{3.4}$$

where the expectation in (3.4) is calculated with respect to the marginal (predictive) distribution of M.

Next, we turn to the estimation of β in this problem where θ is unknown. It can be shown that with respect to the same loss function L_1 as before, as given by (2.3), the Bayes estimator now becomes

$$\hat{\beta} = \frac{r_1-2}{s_1} \cdot \frac{s_2}{r_2+1} \quad . \tag{3.5}$$

It is interesting to note that this estimator is the same as the estimator $\hat{\beta}$ given in (2.6) when the value of θ was known, except that the known value $\theta = \theta_0$ is now replaced in that expression by the estimate

$$\frac{r_2+1}{s_2} = \frac{E'(\theta^2)}{E'(\theta)} \quad , \tag{3.6}$$

where the symbol E' in (3.6) indicates that the expectation is to be calculated with respect to the posterior distribution of θ.

It can be shown that the Bayes risk of the estimator given by (3.5) is

$$E\left[\frac{r_1+r_2-1}{(r_1-1)(r_2+1)}\right] = E\left[\frac{r+r_0+n-1}{(r+M-1)(r_0+n-M+1)}\right] \quad . \tag{3.7}$$

As before, the expectation in (3.7) is to be calculated with respect to the marginal distribution of M.

Finally, suppose that we are interested in estimating both β and θ in this problem, and that the loss function is of the form

$$L(\hat{\beta},\hat{\theta};\beta,\theta) = \lambda_1 L_1(\hat{\beta},\beta) + \lambda_2 L_1(\hat{\theta},\theta), \tag{3.8}$$

where λ_1 and λ_2 are given positive constants. Then the Bayes estimators $\hat{\beta}$ and $\hat{\theta}$ are again as given in (3.5) and (3.3), and the overall Bayes risk ρ is simply

$$\rho = \lambda_1[\text{Risk given by (3.7)}] + \lambda_2[\text{Risk given by (3.4)}]. \quad (3.9)$$

4. Optimal design

Suppose now that the experimenter has to pay a cost for each item tested. In general, this cost will depend on the tampering point x and on whether or not the observation is actually tampered. Under these conditions, the experimenter desires to choose an optimal design for the estimation of the unknown parameters β and θ by choosing the n tampering points $x_1,\ldots,x_n$ so that the total risk (the sum of the Bayes risk due to estimation error and the expected cost of using the tampering points) is a minimum.

In many problems of optimal experimental design, it is difficult to obtain a closed-form solution to this minimization problem unless a simple closed-form expression for the Bayes risk ρ given by (3.9) is available. In our problem, such an expression is not available because it is difficult to determine the expectations in (3.4) and (3.7) as explicit functions of $x_1,\ldots,x_n$. In this section, we will show that despite this difficulty, we can obtain simple, explicit optimal designs for various types of tampering costs. The basic property that we

shall use is that the Bayes risk ρ is of the form

$$\rho = E[h(M)] , \tag{4.1}$$

where $h(M)$ is an explicit, known function of M. Indeed, it follows from (3.9) that

$$h(M) = \frac{1}{r_0+n-M+1} \left[\frac{\lambda_1(r+r_0+n-1)}{r+m-1} + \lambda_2 \right] . \tag{4.2}$$

Suppose first, as a simple example, that the cost of an observation depends only on whether or not it turns out to be tampered, and not on the value of the tampering point. Suppose that the cost of each untampered observation is $v_1 \geq 0$ and the cost of each tampered observation is $v_2 \geq 0$. Then for any tampering points $x_1,\ldots,x_n$, the cost of the n observations will be

$$c(M) = (n-M)v_1 + Mv_2 = nv_1 + (v_2-v_1)M . \tag{4.3}$$

Thus, the total risk R is given by

$$R = E[h(M) + c(M)] . \tag{4.4}$$

In (4.4) we have represented R as the expectation of an explicitly known function of M. Therefore, among all possible

distributions of M, R will be minimized when the distribution of M assigns probability 1 to the integer m_0 that actually minimizes the function $h(M) + c(M)$; i.e.,

$$h(m_0) + c(m_0) = \min_{0,1,\ldots,n} [h(i) + c(i)] . \tag{4.5}$$

Can this degenerate distribution of M actually be obtained from some particular choice of the tampering points $x_1,\ldots,x_n$? The answer is yes: We choose m_0 tampering points at $x = 0$, so that these observations are tampered immediately, and the remaining $n - m_0$ tampering points at $x = \infty$, so that these observations are never tampered. Thus, under the optimal design the experimenter never leaves to chance whether or not an observation will be tampered.

The cost structure we have just considered is random in the sense that the cost of an observation is not fixed in advance but depends on whether or not the observation turns out to be tampered. We shall now assume that the cost $c(x)$ of each observation is fixed in advance and depends only on the tampering point x. For the optimal design, we need to choose the tampering points $x_1,\ldots,x_n$ to minimize

$$R = E[h(M)] + \sum_{i=1}^{n} c(x_i) . \tag{4.6}$$

For any given tampering point x, let

$$p(x) = E[Pr(T > x|\theta)] = E(e^{-x\theta}) , \tag{4.7}$$

where the expectation is evaluated with respect to the prior distribution of θ. In other words, $p(x)$ is the prior probability that an observation will be tampered when the tampering point x is used. It follows that $E(M) = \sum_{i=1}^{n} p(x_i)$.

Now suppose that the cost function $c(x)$ has the special form

$$c(x) = a + bp(x) . \tag{4.8}$$

Then R, as given by (4.6), reduces

$$R = E[h(M) + na + bM] . \tag{4.9}$$

It follows that the optimal design in this problem will be the same as that based on the risk function in the preceding example defined by (4.3) and (4.4), with $a = \nu_1$ and $b = \nu_2 - \nu_1$.

In both the first example that we presented in this section, in which the cost of an observation was random, and in the second example, in which the cost of an observation was fixed, the optimal design was found to use only the two tampering points $x = 0$ and $x = \infty$. In fact, as we will now explain, there is

a wide class of cost functions $c(x)$ for which the optimal design has this property.

For any prior distribution of θ on the parameter space $0 < \theta < \infty$, the function $p(x)$ defined by (4.7) will be a strictly decreasing function of x for $x \geq 0$. Hence, any cost function $c(x)$ can be expressed in the form

$$c(x) = c^*[p(x)], \tag{4.10}$$

where $c^*(p)$ is defined for $0 \leq p \leq 1$ and has the interpretation that it is the cost of choosing a tampering point x for which the probability that the resulting observation will be tampered is p. Suppose now that the cost function $c(x)$ yields a function $c^*(p)$ satisfying the following condition:

$$c^*(p) \geq pc^*(1) + (1-p)c^*(0) \quad \text{for } 0 < p < 1 . \tag{4.11}$$

Then it can be shown that the total risk is minimized by a design that uses only the tampering points $x = 0$ and $x = \infty$. It is a corollary of this result that if $c^*(p)$ is a concave function on the interval $0 \leq p \leq 1$, then there is an optimal design using only the points $x = 0$ and $x = \infty$.

In conclusion, the special nature of the particular examples presented in this section should be emphasized. They are special because in each case the optimal design can be determined simply

from the function $h(M)$, without any further consideration of the predictive distribution of M. For more general cost functions, this avenue of solution will not be open, and the optimal designs will involve tampering points x with $0 < x < \infty$.

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